

AN OPTIMAL DIFFERENTIABLE SPHERE THEOREM FOR COMPLETE MANIFOLDS *

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Abstract

A new differentiable sphere theorem is obtained from the view of submanifold geometry. An important scalar is defined by the scalar curvature and the mean curvature of an oriented complete submanifold M^n in a space form $F^{n+p}(c)$ with $c \geq 0$. Making use of the Hamilton-Brendle-Schoen convergence result for Ricci flow and the Lawson-Simons-Xin formula for the nonexistence of stable currents, we prove that if the infimum of this scalar is positive, then M is diffeomorphic to S^n . We then introduce an intrinsic invariant $I(M)$ for oriented complete Riemannian n -manifold M via the scalar, and prove that if $I(M) > 0$, then M is diffeomorphic to S^n . It should be emphasized that our differentiable sphere theorem is optimal for arbitrary $n(\geq 2)$.

1. Introduction

The investigation of curvature and topology of Riemannian manifolds or submanifolds is one of the main stream in global differential geometry. In 1898, Hadamard [9] proved a classical sphere theorem which says that any oriented compact surface with positive Gaussian curvature in R^3 must be diffeomorphic a sphere. It was seen from the Gauss-Bonnet theorem that Hadamard's sphere theorem could be extended to the cases of compact Riemannian surfaces with positive curvature. A natural problem is stated as follows.

Problem 1.1. *Is it possible to generalize the Hadamard sphere theorem for compact Riemannian surfaces to higher dimensional cases?*

In 1951, Rauch [16] first proved a topological sphere theorem for positive pinched compact manifolds. During the past six decades, there are many important progresses on topological and differentiable pinching problems for Riemannian manifolds. The most famous topological sphere theorem is Berger-Klingenberg's quarter pinching theorem, which has been improved by many geometers [2, 4, 17, 23]. Recently Brendle and Schoen [6] obtained a classification of compact and simply connected manifolds with weakly $1/4$ -pinched curvatures. Consequently, they obtained the following striking result.

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Theorem A. *Let M be an n -dimensional complete and simply connected Riemannian manifold such that $1/4 \leq K_M \leq 1$. Then M is either diffeomorphic to S^n , or isometric to a compact rank one symmetric space (CROSS).*

Since the dimension of a complex projective space is always even, Brendle and Schoen's differentiable sphere theorem is optimal for even dimensional cases. More recently Petersen and Tao [14] have improved Brendle and Schoen's pinching constant in Theorem A to $\frac{1}{4} - \varepsilon_n$, where ε_n is a positive constant depending only on n . However, Petersen and Tao's pinching constant is not yet optimal for odd dimensional cases.

Let M^n be an $n(\geq 2)$ -dimensional submanifold in an $(n+p)$ -dimensional Riemannian manifold N^{n+p} . Denote by H and S the mean curvature and the squared length of the second fundamental form of M respectively. Using nonexistence for stable currents on compact submanifolds of a sphere and the generalized Poincaré conjecture for dimension $n(\geq 5)$ proved by Smale, Lawson and Simons [13] proved that if $M^n(n \geq 5)$ is an oriented compact submanifold in S^{n+p} , and if $S < 2\sqrt{n-1}$, then M is homeomorphic to a sphere.

Let $F^{n+p}(c)$ be an $(n+p)$ -dimensional simply connected space form with nonnegative constant curvature c . Putting

$$\alpha(n, H, c) = nc + \frac{n^3}{2(n-1)}H^2 - \frac{n(n-2)}{2(n-1)}\sqrt{n^2H^4 + 4(n-1)cH^2},$$

we have $\min_H \alpha(n, H, c) = 2\sqrt{n-1}c$. Motivated by a rigidity theorem in [20, 21], Shiohama and Xu [18] improved Lawson-Simons' result and proved the following

Theorem B. *Let $M^n(n \geq 4)$ be an oriented complete submanifold in $F^{n+p}(c)$ with $c \geq 0$. Suppose that $\sup_M(S - \alpha(n, H, c)) < 0$. Then M is homeomorphic to a sphere.*

The following differentiable sphere theorem for hypersurfaces follows from the convergence results for the mean curvature flow and parabolic flow due to Huisken [11] and Andrews [1], respectively.

Theorem C. *Let M^n be an n -dimensional oriented closed hypersurface in $F^{n+1}(c)$ with $c > 0$. If $S < 2c + \frac{n^2H^2}{n-1}$, then M is diffeomorphic to S^n .*

Recently, Xu and Zhao [22] proved a differentiable sphere theorem for submanifolds of a sphere with codimension $p(\geq 1)$.

Theorem D. *Let M^n be an $n(\geq 4)$ -dimensional oriented complete submanifold in $F^{n+p}(c)$ with $c > 0$. Then*

- (i) *if $4 \leq n \leq 6$ and $\sup_M S < 2\sqrt{n-1}c$, then M is diffeomorphic to S^n ,*
- (ii) *if $n \geq 7$ and $S < 2\sqrt{2}c$, then M is diffeomorphic to S^n .*

Motivated by Theorems B, C and D, we propose the following differentiable pinching problem.

Problem 1.2. *Let M^n be an oriented complete submanifold in $F^{n+p}(c)$ with $c \geq 0$. Suppose that $\sup_M \left(S - \frac{n^2H^2}{n-1} - 2c \right) < 0$. Is it possible to prove that M must be diffeomorphic to S^n ?*

The purpose of the present article is to solve Problems 1.1 and 1.2, and prove some new differentiable pinching theorems for complete submanifolds and Riemannian manifolds via Ricci flow and stable currents. More precisely, we obtain the following

Theorem 1.1. *Let M^n be an n -dimensional complete submanifold in an $(n+p)$ -dimensional Riemannian manifold N^{n+p} . Denote by $\overline{K}(\pi)$ the sectional curvature of N for tangent 2-plane $\pi(\subset T_x N)$ at point $x \in N$. Set $\overline{K}_{\max} := \max_{\pi \subset T_x N} \overline{K}(\pi)$, $\overline{K}_{\min} := \min_{\pi \subset T_x N} \overline{K}(\pi)$. If $S < \frac{8}{3} \left(\overline{K}_{\min} - \frac{1}{4} \overline{K}_{\max} \right) + \frac{n^2 H^2}{n-1}$, then M is diffeomorphic to a space form. In particular, if M is simply connected, then M is diffeomorphic to S^n or R^n .*

Theorem 1.2. *Let M^n be an n -dimensional oriented complete submanifold in $F^{n+p}(c)$ with $c \geq 0$. If*

$$\lambda(M) := \sup_M \left(S - \frac{n^2 H^2}{n-1} - 2c \right) < 0,$$

then M is diffeomorphic to S^n .

We shall show in Example 4.1 that Theorem 1.2 is optimal for arbitrary $n(\geq 2)$ and p . It follows from the Gauss equation that the pinching condition in Theorem 1.2 is equivalent to

$$\mu(M) := -\lambda(M) = \inf_M \left[R - \frac{n^2(n-2)}{n-1} H^2 - (n+1)(n-2)c \right] > 0,$$

where R is the scalar curvature of M .

For a complete Riemannian n -manifold M^n , we set $\mathcal{C} := \{\varphi; \varphi : M \rightarrow F^{n+p}(c) \text{ is an isometric embedding for some constant } c \geq 0 \text{ and positive integer } p\}$ and $\mathcal{D} := \{\varphi; \varphi : M \rightarrow R^{n+p} \text{ is an isometric embedding for some positive integer } p\}$. With the aid of the Nash embedding theorem, we get $\mathcal{C} \supset \mathcal{D} \neq \emptyset$. We define two intrinsic invariants $I(M)$ and $I_0(M)$ by

$$I(M) := \sup_{\varphi \in \mathcal{C}} \mu(M, \varphi) := \sup_{\varphi \in \mathcal{C}} \inf_M \left[R - \frac{n^2(n-2)}{n-1} H^2 - (n+1)(n-2)c \right],$$

$$I_0(M) := \sup_{\varphi \in \mathcal{D}} \mu(M, \varphi) := \sup_{\varphi \in \mathcal{D}} \inf_M \left[R - \frac{n^2(n-2)}{n-1} H^2 \right].$$

Notice that $I(M) \geq I_0(M)$. We shall prove

Theorem 1.3. *Let M^n be an oriented complete Riemannian n -manifold. If $I(M) > 0$, then M is diffeomorphic to S^n . In particular, if $I_0(M) > 0$, then M is diffeomorphic to S^n .*

Remark 1.1. In the case $n = 2$, Theorem 1.3 is reduce to the Hadamard sphere theorem for compact Riemannian surfaces. We shall give an example (Example 4.2) to show that our differentiable sphere theorem for Riemannian manifolds is optimal for arbitrary $n(\geq 2)$.

Furthermore, we obtain some other differentiable pinching theorems for complete submanifolds in Riemannian manifolds, which extend the sphere theorems due to Huisken, Xu and Zhao [11, 22].

2. Notation and lemmas

Let M^n be an $n(\geq 2)$ -dimensional submanifold in an $(n+p)$ -dimensional Riemannian manifolds N^{n+p} . We shall make use of the following convention on the range of indices.

$$1 \leq A, B, C, \dots \leq n+p; \quad 1 \leq i, j, k, \dots \leq n; \quad n+1 \leq \alpha, \beta, \gamma, \dots \leq n+p.$$

For an arbitrary fixed point $x \in M \subset N$, we choose an orthonormal local frame field $\{e_A\}$ in N^{n+p} such that e_i 's are tangent to M . Denote by $\{\omega_A\}$ the dual frame field of $\{e_A\}$. Let Rm and \overline{Rm} be the Riemannian curvature tensors of M and N respectively, and h the second fundamental form of M . Then

$$\begin{aligned} Rm &= \sum_{i,j,k,l} R_{ijkl} \omega_i \otimes \omega_j \otimes \omega_k \otimes \omega_l, \\ \overline{Rm} &= \sum_{A,B,C,D} \overline{R}_{ABCD} \omega_A \otimes \omega_B \otimes \omega_C \otimes \omega_D, \\ h &= \sum_{\alpha,i,j} h_{ij}^\alpha \omega_i \otimes \omega_j \otimes e_\alpha, \\ R_{ijkl} &= \overline{R}_{ijkl} + \sum_{\alpha} (h_{ik}^\alpha h_{jl}^\alpha - h_{il}^\alpha h_{jk}^\alpha). \end{aligned} \tag{1}$$

The squared norm S of the second fundamental form and the mean curvature H of M are given by

$$S := \sum_{\alpha,i,j} (h_{ij}^\alpha)^2, \quad H := \left| \frac{1}{n} \sum_{\alpha,i} h_{ii}^\alpha e_\alpha \right|.$$

Denote by $K(\pi)$ the sectional curvature of M for tangent 2-plane $\pi(\subset T_x M)$ at point $x \in M$, $\overline{K}(\pi)$ the sectional curvature of N for tangent 2-plane $\pi(\subset T_x N)$ at point $x \in N$. Set $\overline{K}_{\min} := \min_{\pi \subset T_x N} \overline{K}(\pi)$, $\overline{K}_{\max} := \max_{\pi \subset T_x N} \overline{K}(\pi)$.

The Lawson-Simons-Xin non-existence theorem [13, 19] for stable currents in a compact Riemannian manifold M isometrically immersed into $F^{n+p}(c)$ is employed to eliminate the homology groups $H_q(M; Z)$ for $0 < q < n$.

Lemma 2.1. *Let M^n be a compact submanifold in $F^{n+p}(c)$ with $c \geq 0$. Assume that*

$$\sum_{k=q+1}^n \sum_{i=1}^q [2|h(e_i, e_k)|^2 - \langle h(e_i, e_i), h(e_k, e_k) \rangle] < q(n-q)c$$

holds for any orthonormal basis $\{e_i\}$ of M_x at any point $x \in M$, where q is an integer satisfying $0 < q < n$. Then there does not exist any stable q -currents. Moreover,

$$H_q(M; Z) = H_{n-q}(M; Z) = 0,$$

where $H_i(M; \mathbb{Z})$ is the i -th homology group of M with integer coefficients.

The following convergence result for the Ricci flow, initialed by Brendle and Schoen [5], was finally obtained by Brendle [3].

Lemma 2.2. *Let (M, g_0) be a compact Riemannian manifold of dimension $n(\geq 4)$. Assume that*

$$R_{1313} + \lambda^2 R_{1414} + R_{2323} + \lambda^2 R_{2424} - 2\lambda R_{1234} > 0 \quad (2)$$

for all orthonormal four-frames $\{e_1, e_2, e_3, e_4\}$ and all $\lambda \in [-1, 1]$. Then the normalized Ricci flow with initial metric g_0

$$\frac{\partial}{\partial t} g(t) = -2\text{Ric}_{g(t)} + \frac{2}{n} r_{g(t)} g(t),$$

exists for all time and converges to a constant curvature metric as $t \rightarrow \infty$. Here $r_{g(t)}$ denotes the mean value of the scalar curvature of $g(t)$.

3. Sphere Theorem in dimension three

Using Lemma 2.1 and the assumption for S , we obtain the following

Theorem 3.1. *Let M^3 be a 3-dimensional oriented compact submanifold in a simply connected space form $F^{3+p}(c)$ with nonnegative constant curvature c . If $S < 2c + \frac{9}{2}H^2$, then M is diffeomorphic to S^3 .*

Proof. We observe that

$$\begin{aligned} & \sum_{k=q+1}^3 \sum_{i=1}^q [2|h(e_i, e_k)|^2 - \langle h(e_i, e_i), h(e_k, e_k) \rangle] \\ &= 2 \sum_{\alpha} \sum_{k=q+1}^3 \sum_{i=1}^q (h_{ik}^{\alpha})^2 - \sum_{\alpha} \sum_{k=q+1}^3 \sum_{i=1}^q h_{ii}^{\alpha} h_{kk}^{\alpha} \\ &= \sum_{\alpha} \left[2 \sum_{k=q+1}^3 \sum_{i=1}^q (h_{ik}^{\alpha})^2 - \left(\sum_{i=1}^q h_{ii}^{\alpha} \right) \left(\sum_{i=1}^3 h_{ii}^{\alpha} - \sum_{i=1}^q h_{ii}^{\alpha} \right) \right]. \end{aligned} \quad (3)$$

Setting

$$S_{\alpha} := \sum_{i,j=1}^3 (h_{ij}^{\alpha})^2, \quad T_{\alpha} := \sum_{i=1}^3 h_{ii}^{\alpha}, \quad \tilde{S}_{\alpha} := \sum_{i=1}^3 (h_{ii}^{\alpha})^2,$$

we have

$$S = \sum_{\alpha} S_{\alpha}, \quad 9H^2 = \sum_{\alpha} T_{\alpha}^2,$$

and

$$qr\tilde{S}_{\alpha} = qr \sum_{i=1}^q (h_{ii}^{\alpha})^2 + qr \sum_{k=q+1}^3 (h_{kk}^{\alpha})^2 \geq r \left(\sum_{i=1}^q h_{ii}^{\alpha} \right)^2 + q \left(\sum_{k=q+1}^3 h_{kk}^{\alpha} \right)^2, \quad (4)$$

where $r := 3 - q$. Inserting

$$T_\alpha - \sum_{i=1}^q h_{ii}^\alpha = \sum_{k=q+1}^3 h_{kk}^\alpha,$$

into the right hand side of (4), we get

$$3\left(\sum_{i=1}^q h_{ii}^\alpha\right)^2 - 2qT_\alpha \sum_{i=1}^q h_{ii}^\alpha + qT_\alpha^2 - qr\tilde{S}_\alpha \leq 0. \quad (5)$$

Set

$$Z_\alpha := -\left(\sum_{i=1}^q h_{ii}^\alpha\right)\left(T_\alpha - \sum_{i=1}^q h_{ii}^\alpha\right).$$

It follows from (5) that

$$3Z_\alpha + (r - q)T_\alpha \sum_{i=1}^q h_{ii}^\alpha + qT_\alpha^2 - qr\tilde{S}_\alpha \leq 0. \quad (6)$$

Making use of the relations

$$\sum_{i=1}^3 \left(h_{ii}^\alpha - \frac{T_\alpha}{3}\right)^2 = \tilde{S}_\alpha - \frac{T_\alpha^2}{3}, \quad \sum_{i=1}^3 \left(h_{ii}^\alpha - \frac{T_\alpha}{3}\right) = 0, \quad \sum_{i=1}^q \left(h_{ii}^\alpha - \frac{T_\alpha}{3}\right) + \frac{q}{3}T_\alpha = \sum_{i=1}^q h_{ii}^\alpha,$$

and setting $\tilde{h}_{ii}^\alpha := h_{ii}^\alpha - \frac{T_\alpha}{3}$, we obtain

$$\tilde{S}_\alpha - \frac{T_\alpha^2}{3} \geq \frac{1}{q} \left(\sum_{i=1}^q \tilde{h}_{ii}^\alpha\right)^2 + \frac{1}{r} \left(\sum_{k=q+1}^3 \tilde{h}_{kk}^\alpha\right)^2 = \left(\frac{1}{q} + \frac{1}{r}\right) \left[\sum_{i=1}^q \left(h_{ii}^\alpha - \frac{T_\alpha}{3}\right)\right]^2. \quad (7)$$

Therefore we find

$$\left|\sum_{i=1}^q \left(h_{ii}^\alpha - \frac{T_\alpha}{3}\right)\right| \leq \sqrt{\frac{qr}{3} \left(\tilde{S}_\alpha - \frac{T_\alpha^2}{3}\right)}. \quad (8)$$

This together with (6) implies

$$Z_\alpha \leq \frac{qr}{3} \tilde{S}_\alpha - \left[\frac{q(r-q)}{9} + \frac{q}{3}\right] T_\alpha^2 + \frac{|r-q|}{3} |T_\alpha| \sqrt{\frac{qr}{3} \left(\tilde{S}_\alpha - \frac{T_\alpha^2}{3}\right)}. \quad (9)$$

From (3), (9) and the fact $qr = 2$ and $|r - q| = 1$, we obtain

$$\begin{aligned} & \sum_{k=q+1}^3 \sum_{i=1}^q [2|h(e_i, e_k)|^2 - \langle h(e_i, e_i), h(e_k, e_k) \rangle] - qrc \\ & \leq \sum_{\alpha} \left[S_\alpha - \frac{\tilde{S}_\alpha}{3} - \frac{4}{9} T_\alpha^2 + \frac{|T_\alpha|}{3} \sqrt{\frac{2}{3} \left(\tilde{S}_\alpha - \frac{T_\alpha^2}{3}\right)} \right] - 2c \\ & \leq S - 4H^2 - 2c - \frac{1}{3} \sum_{\alpha} \tilde{S}_\alpha + \sum_{\alpha} \left[\frac{T_\alpha^2}{18} + \frac{1}{3} \left(\tilde{S}_\alpha - \frac{T_\alpha^2}{3}\right) \right] \\ & = S - \frac{9}{2} H^2 - 2c \end{aligned} \quad (10)$$

Then under the assumption, we obtain

$$\sum_{k=q+1}^3 \sum_{i=1}^q [2|h(e_i, e_k)|^2 - \langle h(e_i, e_i), h(e_k, e_k) \rangle] - qrc < 0. \quad (11)$$

Suppose that $\pi_1(M) \neq 0$. Since M is compact, it follows from a classical theorem due to Cartan and Hadamard that there exists a minimal closed geodesic in any non-trivial homotopy class in $\pi_1(M)$. However, combining Lemma 2.1 and (11) we know that there does not exist any stable integral currents on M . This contradicts the hypothesis. Therefore, $\pi_1(M) = 0$.

It follows from Proposition 2.1 in [18] and the assumption for S that

$$\begin{aligned} Ric_M(X) &\geq \frac{2}{3} \left[3c + 6H^2 - S - \frac{3}{\sqrt{6}} H(S - 3H^2)^{1/2} \right] \\ &= \frac{2}{3} \left[(3c + \frac{27}{4}H^2 - \frac{3}{2}S) + \frac{3}{4}H^2 + \frac{1}{2}(S - 3H^2) - \frac{3}{\sqrt{6}} H(S - 3H^2)^{1/2} \right] > 0 \end{aligned}$$

holds for any unit vector $X \in T_x M$. By Hamilton's convergence result for Ricci flow in three dimensions [10], it follows that M is diffeomorphic to a 3-dimensional spherical space form. This completes the proof of Theorem 3.1.

Corollary 3.1. *Let M be a 3-dimensional oriented compact submanifold in the unit sphere S^{3+p} . Suppose that $H \geq \frac{2}{3}\sqrt{\sqrt{2}-1}$. If $S < 2\sqrt{2}$, then M is diffeomorphic to S^3 .*

Proof. By a direct computation, we get

$$S < 2 + \frac{9}{2}H^2.$$

By Theorem 3.1, we see that M is diffeomorphic to S^3 . This proves Corollary 3.1.

Up to now, the following problem proposed by Lawson and Simons [13] is still open.

Problem 3.1. *Let M be a 3-dimensional oriented compact submanifold in the unit sphere S^{3+p} . Suppose that $S < 2\sqrt{2}$. Can one prove that M must be diffeomorphic to S^3 ?*

4. Differentiable sphere theorem in higher dimensions

The following lemma will be used in the proof of our theorems.

Lemma 4.1. *Let M^n be an n -dimensional submanifold in an $(n+p)$ -dimensional Riemannian manifold N^{n+p} , and π a tangent 2-plane on $T_x M$ at point $x \in M$. Choose an orthonormal two-frame $\{e_1, e_2\}$ at x such that $\pi = \text{span}\{e_1, e_2\}$. Then*

$$K(\pi) \geq \frac{1}{2} \left(2\bar{K}_{\min} + \frac{n^2 H^2}{n-1} - S \right) + \sum_{\alpha=n+1}^{n+p} \sum_{j>i, (i,j) \neq (1,2)} (h_{ij}^\alpha)^2. \quad (12)$$

Proof. We extend the orthonormal two-frame $\{e_1, e_2\}$ to $\{e_1, \dots, e_{n+p}\}$ such that e_i 's are tangent to M . Setting $S_\alpha := \sum_{i,j=1}^n (h_{ij}^\alpha)^2$, we have

$$\left(\sum_{i=1}^n h_{ii}^\alpha\right)^2 = (n-1) \left[\sum_{i=1}^n (h_{ii}^\alpha)^2 + \sum_{i \neq j} (h_{ij}^\alpha)^2 + \frac{(\sum_{i=1}^n h_{ii}^\alpha)^2}{n-1} - S_\alpha \right]. \quad (13)$$

Note that

$$\begin{aligned} \left(\sum_{i=1}^n h_{ii}^\alpha\right)^2 &\leq (n-1) \left[(h_{11}^\alpha + h_{22}^\alpha)^2 + \sum_{i>2} (h_{ii}^\alpha)^2 \right] \\ &= (n-1) \left[\sum_{i=1}^n (h_{ii}^\alpha)^2 + 2h_{11}^\alpha h_{22}^\alpha \right]. \end{aligned}$$

This together with (13) implies

$$2h_{11}^\alpha h_{22}^\alpha \geq \sum_{i \neq j} (h_{ij}^\alpha)^2 + \frac{(\sum_{i=1}^n h_{ii}^\alpha)^2}{n-1} - S_\alpha. \quad (14)$$

From the Gauss equation and (14) we get

$$\begin{aligned} K(\pi) &= \bar{R}_{1212} + \sum_{\alpha=n+1}^{n+p} [h_{11}^\alpha h_{22}^\alpha - (h_{12}^\alpha)^2] \\ &\geq \sum_{\alpha=n+1}^{n+p} \left[\sum_{j>2} (h_{1j}^\alpha)^2 + \sum_{j>2} (h_{2j}^\alpha)^2 + \frac{1}{2} \sum_{i \neq j>2} (h_{ij}^\alpha)^2 \right] + \frac{1}{2} \left(\frac{n^2 H^2}{n-1} - S \right) + \bar{K}_{\min} \\ &\geq \frac{1}{2} \left(2\bar{K}_{\min} + \frac{n^2 H^2}{n-1} - S \right) + \sum_{\alpha=n+1}^{n+p} \sum_{j>i, (i,j) \neq (1,2)} (h_{ij}^\alpha)^2. \end{aligned} \quad (15)$$

This proves Lemma 4.1.

Lemma 4.2. *Let M^n be an n -dimensional complete submanifold in an $(n+p)$ -dimensional Riemannian manifold N^{n+p} . If $\sup_M \left(S - 2\bar{K}_{\min} - \frac{n^2 H^2}{n-1} \right) < 0$, then M is compact.*

Proof. From the assumption and Lemma 4.1, it follows that there exists an $\varepsilon > 0$ such that $K_M \geq \varepsilon$. By the Bonnet-Myers's theorem, we know that M is compact. This completes the proof.

Theorem 4.1. *Let (M, g_0) be an $n(\geq 4)$ -dimensional complete submanifold in an $(n+p)$ -dimensional Riemannian manifold N^{n+p} . If $\sup_M \left[S - \frac{8}{3} \left(\bar{K}_{\min} - \frac{1}{4} \bar{K}_{\max} \right) - \frac{n^2 H^2}{n-1} \right] < 0$, then the normalized Ricci flow with initial metric g_0*

$$\frac{\partial}{\partial t} g(t) = -2\text{Ric}_{g(t)} + \frac{2}{n} r_{g(t)} g(t),$$

exists for all time and converges to a constant curvature metric as $t \rightarrow \infty$. Moreover, M is diffeomorphic to a space form. In particular, if M is simply connected, then M is diffeomorphic to S^n .

Proof. By Lemma 4.2, it follows that M is compact. When $n \geq 4$, suppose $\{e_1, e_2, e_3, e_4\}$ is an orthonormal four-frame and $\lambda \in R$. From the Gauss equation (1) and Berger's inequality we have

$$\begin{aligned} |R_{1234}| &= |\bar{R}_{1234} + \sum_{\alpha} (h_{13}^{\alpha} h_{24}^{\alpha} - h_{14}^{\alpha} h_{23}^{\alpha})| \\ &\leq \frac{2}{3}(\bar{K}_{\max} - \bar{K}_{\min}) + \sum_{\alpha} |h_{13}^{\alpha} h_{24}^{\alpha} - h_{14}^{\alpha} h_{23}^{\alpha}|. \end{aligned} \quad (16)$$

This together with Lemma 4.1 implies

$$\begin{aligned} &R_{1313} + \lambda^2 R_{1414} + R_{2323} + \lambda^2 R_{2424} - 2\lambda R_{1234} \\ &\geq (1 + \lambda^2) \left(2\bar{K}_{\min} + \frac{n^2 H^2}{n-1} - S \right) \\ &\quad + \sum_{\alpha} \sum_{i < j, (i,j) \neq (1,3)} (h_{ij}^{\alpha})^2 + \sum_{\alpha} \sum_{i < j, (i,j) \neq (2,3)} (h_{ij}^{\alpha})^2 \\ &\quad + \lambda^2 \left[\sum_{\alpha} \sum_{i < j, (i,j) \neq (1,4)} (h_{ij}^{\alpha})^2 + \sum_{\alpha} \sum_{i < j, (i,j) \neq (2,4)} (h_{ij}^{\alpha})^2 \right] \\ &\quad - 2|\lambda| \left[\frac{2}{3}(\bar{K}_{\max} - \bar{K}_{\min}) + \sum_{\alpha} |h_{13}^{\alpha} h_{24}^{\alpha} - h_{14}^{\alpha} h_{23}^{\alpha}| \right] \\ &\geq (1 + \lambda^2) \left[\frac{8}{3} \left(\bar{K}_{\min} - \frac{1}{4} \bar{K}_{\max} \right) + \frac{n^2 H^2}{n-1} - S \right] \\ &\quad + \sum_{\alpha} [(h_{24}^{\alpha})^2 + \lambda^2 (h_{13}^{\alpha})^2 + (h_{14}^{\alpha})^2 + \lambda^2 (h_{23}^{\alpha})^2] \\ &\quad - 2|\lambda| |h_{13}^{\alpha} h_{24}^{\alpha}| - 2|\lambda| |h_{14}^{\alpha} h_{23}^{\alpha}| \\ &\geq (1 + \lambda^2) \left[\frac{8}{3} \left(\bar{K}_{\min} - \frac{1}{4} \bar{K}_{\max} \right) + \frac{n^2 H^2}{n-1} - S \right] \\ &> 0. \end{aligned} \quad (17)$$

It follows from Lemma 2.2 that M is diffeomorphic to a space form. In particular, if M is simply connected, then M is diffeomorphic to S^n . This completes the proof Theorem 4.1.

Proof of Theorem 1.1. By the assumption, we have

$$\begin{aligned} S &< \frac{8}{3} \left(\bar{K}_{\min} - \frac{1}{4} \bar{K}_{\max} \right) + \frac{n^2 H^2}{n-1} \\ &\leq 2\bar{K}_{\min} + \frac{n^2 H^2}{n-1}. \end{aligned} \quad (18)$$

So

$$2\bar{K}_{\min} + \frac{n^2 H^2}{n-1} - S > 0.$$

This together with Lemma 4.1 implies $K_M > 0$.

When M is non-compact, a theorem due to Cheeger-Gromoll-Meyer [7, 8] says that M must be diffeomorphic to R^n .

When M is compact, we consider the following cases: (i) If $n = 2$, it follows from the fact $K_M > 0$ that M is diffeomorphic to S^2 or RP^2 . (ii) If $n = 3$, Hamilton's theorem [10] shows that M is diffeomorphic to a spherical space form. (iii) If $n \geq 4$, the assertion follows from Theorem 4.1. In particular, when M is simply connected, we conclude that M must be diffeomorphic to S^n or R^n . This completes the proof of Theorem 1.1.

Theorem 4.2. *Let M^n be an $n(\geq 4)$ -dimensional oriented complete submanifold in an $(n + p)$ -dimensional simply connected space form $F^{n+p}(c)$ with nonnegative constant curvature c . If $\sup_M \left(S - \frac{n^2 H^2}{n-1} \right) < 2c$, then M is diffeomorphic to S^n .*

Proof. It is easy to see that

$$\begin{aligned} \alpha(n, H, c) &= nc + \frac{n^3}{2(n-1)} H^2 - \frac{n(n-2)}{2(n-1)} \sqrt{n^2 H^4 + 4(n-1)c H^2} \\ &\geq nc + \frac{n^3}{2(n-1)} H^2 - \frac{n(n-2)}{2(n-1)} \left[\frac{n H^2}{2} + \frac{n^2 H^2 + 4(n-1)c}{2n} \right] \\ &= 2c + \frac{n^2 H^2}{n-1}. \end{aligned} \tag{19}$$

It follows from Theorem B that M is a topological sphere.

On the other hand, we see from Theorem 4.1 that M is diffeomorphic to a space form. Therefore, M is diffeomorphic to S^n . This proves Theorem 4.2.

Theorem 4.3. *Let M^n be an n -dimensional oriented complete submanifold in an $(n + p)$ -dimensional simply connected space form $F^{n+p}(c)$ with nonnegative constant curvature c . If $S < 2c + \frac{n^2 H^2}{n-1}$, then M is diffeomorphic to S^n or R^n .*

Proof. From Lemma 4.1, we know that $K_M > 0$. When M is non-compact, the assertion follows from the proof of Theorem 1.1.

When M is compact, we consider the following cases: (i) If $n = 2$, from the Gauss-Bonnet theorem we see that the genus of M is zero, and hence M is a topological sphere. Therefore, M is diffeomorphic to S^2 . (ii) If $n \geq 3$, it follows from Theorems 3.1 and 4.2 that M is diffeomorphic to S^n . This completes the proof.

Proof of Theorem 1.2. From Lemma 4.2, we know that M is compact. This together with Theorem 4.3 implies that M is diffeomorphic to S^n . This completes the proof of Theorem 1.2.

The following example shows that the pinching conditions in Theorems 1.2 and 4.3 are the best possible for arbitrary $n(\geq 2)$ and p .

Example 4.1. (i) When $c = 0$, let $M := S^{n-1} \left(\frac{n-1}{nH_0} \right) \times R^1 \subset R^{n+1} \subset R^{n+p}$, where H_0 is a positive constant. Then $H = H_0$ and $S = \frac{n^2 H^2}{n-1}$. (ii) When $c > 0$, without loss of generality,

we only consider the case $c = 1$. Let $M := S^1\left(\frac{1}{\sqrt{1+\lambda^2}}\right) \times S^{n-1}\left(\frac{\lambda}{\sqrt{1+\lambda^2}}\right) \subset S^{n+1} \subset S^{n+p}$, where λ is a positive constant. We have $H = \frac{1}{n}[\lambda - (n-1)\frac{1}{\lambda}]$ and $S = \lambda^2 + (n-1)\frac{1}{\lambda^2}$. Then $S - \frac{n^2 H^2}{n-1} - 2 = \frac{(n-2)}{(n-1)}\lambda^2$. Thus, for any $\varepsilon > 0$ we can find a submanifold $M := S^1\left(\frac{1}{\sqrt{1+\lambda^2}}\right) \times S^{n-1}\left(\frac{\lambda}{\sqrt{1+\lambda^2}}\right) \subset S^{n+p}$ satisfying $S < 2 + \frac{n^2 H^2}{n-1} + \varepsilon$.

Proof of Theorem 1.3. By the assumption, we put $I(M) := \varepsilon_0 > 0$. There exists an isometric embedding $\varphi : M \rightarrow F^{n+p}(c)$ such that

$$\mu(M, \varphi) \geq \frac{1}{2}\varepsilon_0 > 0.$$

Thus $\lambda(M, \varphi) < 0$. It follows from Theorem 1.2 that M is diffeomorphic to S^n . This proves Theorem 1.3.

The following example shows that Theorem 1.3 is optimal for arbitrary $n(\geq 2)$.

Example 4.2. Let $M := S^{n-1}\left(\frac{n-1}{nH_0}\right) \times R^1 \subset R^{n+1} \subset R^{n+p}$, where H_0 is a positive constant. We consider the inclusion $\varphi_0 : M \rightarrow R^{n+p}$. Following Example 4.1, we have

$$\mu(M, \varphi_0) = -\lambda(M, \varphi_0) = 0.$$

This implies $I(M) \geq I_0(M) \geq 0$. Since M is not diffeomorphic to S^n , it follows from Theorem 1.3 that $I_0(M) \leq I(M) \leq 0$. Hence $I(M) = I_0(M) = 0$.

Finally we present the following differentiable sphere theorem for even dimensional submanifolds in a general Riemannian manifold, which is an extension of Theorems 1.2 and 4.3 as well as the sphere theorems due to Huisken, Xu and Zhao [11, 22].

Theorem 4.4. *Let M^n be an even dimensional oriented complete submanifold in an $(n+p)$ -dimensional Riemannian manifold N^{n+p} . Then*

- (i) *if $S < \frac{8}{3}\left(\overline{K}_{\min} - \frac{1}{4}\overline{K}_{\max}\right) + \frac{n^2 H^2}{n-1}$, then M is diffeomorphic to S^n or R^n .*
- (ii) *if $\sup_M \left[S - \frac{8}{3}\left(\overline{K}_{\min} - \frac{1}{4}\overline{K}_{\max}\right) - \frac{n^2 H^2}{n-1}\right] < 0$, then M is diffeomorphic to S^n .*

Proof. (i) It follows from the assumption and Lemma 4.1 that $K_M > 0$. When M is non-compact, it follows from Cheeger-Gromoll-Meyer's soul theorem [7, 8] that M is diffeomorphic to R^n . When M is compact, it's seen from the assumption and Synge's theorem that M is simply connected. This together with Theorem 1.1 implies that M is diffeomorphic to S^n . Therefore, we conclude that M is diffeomorphic to S^n or R^n .

(ii) From the assumption and Lemma 4.2, we see that M is compact. This together with (i) implies that M is diffeomorphic to S^n . This completes the proof of Theorem 4.4.

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